Grade 6 Mathematics Lesson Plan
Let's think about division of fractions
*Revised version

Date: Saturday, June 25, 2016, 10:00 - 10:45
Multipurpose Room
Grade 6 Homeroom 2, 29 students
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1 About the Unit

According to a recent report on the results of the national achievement test, the success rates for "fraction × fraction" and "fraction ÷ fraction" were 91.6% and 91.2%, respectively. On the other hand, the passing rates for questions asking for "explanation for ways of calculation" were 46.0% for "fraction × fraction" and 56.9% for "fraction ÷ fraction." Thus, the report concluded that "students remembering how to calculate procedurally without clearly understanding its meaning" is remains a challenge.

In a 2015 study, Nakamura (Yamanashi University) asked undergraduate students in the College of Education the following question.

> We are making ⅕-kg hamburger patties using 3½ kg of ground meat. How many hamburger patties can we make, and how much ground meat will be left over?

The success rate was 36% and 20% of students did not provide any answer. This problem is a quotitive division problem. It is relatively easy to calculate "3½ ÷ ⅕ = 35/2 = 17½." However, because interpreting the meaning of ½ in the quotient expressed as a mixed number is challenging, it is rarely dealt with in textbooks.

Keeping the above in mind, in this unit, we focused on the meaning of dividing by fractions and also the quotient (fractions). We designed the unit so that students will apply what they have learned in their prior learning of arithmetical calculations and devise their methods of calculating "÷ fraction."

Students first learned the multiplication by whole number multipliers as "(group size) × (number of groups) = (total amount)." When multipliers became decimal numbers or fractions, the meaning of multiplication is expanded to "(referent amount) × (ratio) = (relative amount)." With division, students learned to consider division as the inverse operation of multiplication when divisors are both whole numbers and decimal numbers. They have also experienced the derivation of an algorithm for division with decimal divisors by making use of the property of division, when both the dividend and the divisor are multiplied (or divided) by the same number, the quotient remains unchanged.

With respect fractions, students learned about the meaning of division such as "so many parts of 1 partitioned into a number of equal parts" and "$a \div b = a/b$ (fraction as a quotient)." Students learned about properties of fractions through investigations of their size relationships and used those properties in their study of addition and subtraction calculations with fractions.

In this unit, students think about the meaning of division by fractions and devise ways of carrying out the calculations. This unit complete the study of arithmetic operations with whole numbers, decimal numbers, and fractions.

In teaching this unit, the goal is for students to understand the meaning of division as the inverse operation of "(referent quantity) × (ratio) = (relative quantity)," that is the operation to find either the referent quantity or the ratio. We will first represent problem situations using double number lines to identify the proportional relationship of the quantities. We will set up multiplication equations using □ for the unknown quantity, then finally division equations. In the introduction of the unit, we will use partitive division which determines the referent quantity just as we did with the study of division by decimal numbers. Although it may be more difficult to make sense of the meaning of division, the meaning of the quotient is straightforward and students do not have to deal with the remainder. At the conclusion of the unit, we will treat quotitive division which is used to determine the ratio to deepen students understanding of the meaning of division by fractions.

In addition, we want students to devise ways of calculations on their own. To help students pay attention to the properties of operations and fractions they have previously learned, we intend to provide opportunities for students to look back on their study of division and fractions.

2 Goals of the Unit

- ■ Students will understand the meaning and ways of calculating division by fractions and extend their ability to apply their learning.
- ○ Students will become interested in the meaning and ways of calculating division by fractions, and they try to make connections to their prior learning of calculations and properties of division. (Interest, Eagerness, and Attitude)
- ○ Students can think about ways of calculating division by fractions based on the properties of division and proportional relationship, and they can represent those situations using double number lines and equations. (Mathematical Way of Thinking)
- ○ Students can calculate division by fractions and apply them. (Mathematical Skills)
- ○ Students understand the meaning of division of fractions. (Knowledge and Understanding)

3 Relationship between the Unit and the Research Theme

(1) About the dispositions·abilities we want to nurture in this unit

In the mathematics group, in order to realize lessons in which students create their own mathematics, we utilize lessons that focus on problem solving (*mondai kaiketsu gakushu*). In the learning processes in problem solving lessons, the four dispositions·abilities and "questions" are closely related. (In the proceedings for research open house, the following 4 dispositions·abilities have been identified: (1) dispositions·abilities that are skill/procedural, (2) dispositions·abilities that are cognitive, (3) dispositions·abilities that are affective, and (4) dispositions·abilities that are social.)

In the stage of grasping the learning task, students will have the question, "What have I learned so far?" and put the problem situations from their daily lives onto the mathematical playing field as the first step of problem solving. In the independent problem solving stage of the lesson, students will ask themselves, "Which of what I have learned may be useful in this problem?" and tackle the problem by comparing it to previously solved problems. However, it is not always possible for students to have their own ideas. Thus, their peers will become an important component of their learning. During the comparison and critical reflection stage, students of compare and contrast their own ideas with those of their peers to generate better solutions, approaching the goals of the lesson. In this stage, students will ask about the rationale, commonality, differences, and generalizability. During the reflection stage, students will ask about the merits and extendability of ideas so that they can use what they learned in other situations. In mathematics, we believe the engine for learning is "question." In the process of learning, when one problem is solved, a new "question" arises. "Questions" are continuously generated. What support this learning process is students disposition to tackle problem solving autonomously.

In this unit, we consider "Can we calculate division with fractions in the same way we calculated division?" as the primary "question." In the introductory partitive division situation, in the derivation of the algorithm for "division ÷ division," and in the examination of quotitive division situations, we want to start our study with the question, "Can we use the methods we have learned so far?" This way, we can nurture students' development of the four dispositions･abilities.

(2) About strategies for "lessons in which students feel the values of learning"

In the mathematics group, we consider "values of learning" is realized in experiencing the merits of mathematics. In mathematics "lessons in which students feel the values of learning," students will see the merits of mathematics by comparing and contrasting their own ideas with those of their peers, which in turn generate the next "questions." Thus, those are the lessons in which students experience the merits of mathematics and continuously generate "questions. Thus, in order to realize the sequence of "questions," we must carefully devise the problem situations.

As noted, in "lessons in which students feel the values of learning," "questions" are continuously generated. Mathematics is a subject which builds on students' prior learning. When students construct new knowledge, it is necessary to examine on what prior learning it can build and how to sequence "questions."

The emphases in this unit will be on students understanding the meaning of calculation and the quotient in the contexts involving "÷ fraction" and deriving the algorithm for "fraction ÷ fraction" on their own. Students' prior learning will be the primary support in this process. Therefore, we selected the numbers involved in the learning so that students may have easier time to reflect on their prior learning. It is hoped that students will generate a sequence of questions such as "What is the rationale?" "Is there another way?" or "Can it be done always?" based on their prior learning.

(3) About methods for assessing the quality of individual student's learning

As a strategy to assess the quality of individual student's learning, we will make use of their notebooks. We have been encouraging students to make their notebooks align with the process of problem solving lessons, " grasping the learning task → independent problem solving → comparison and critical reflection → reflection." Thus, in the independent problem solving stage, students will write their own ideas. In the comparison and critical reflection stage, they try to record their peers' ideas. When they do so, instead of simply writing down the answers, we have encouraged them to include the steps and process of getting the answers, using words, pictures, diagrams and mathematical expressions. By writing their learning journal entries, students can organize their ideas, reflect on them deeply and make use of the ideas in new problem situations. This way, students can reflect on the development of their ideas. By checking students' entries in independent problem solving/comparison and critical reflection/learning journal against each other, we will know what ideas students initially had, what challenges they faced, and how their ideas evolved. In this way, we want to assess our efforts to increase the quality of individual student's learning.

4 Unit Plan and Assessment (Total of 11 lessons)

	Goals	Learning Activity	Assessment Standards
① Division of fractions (6 lessons)			
L1	Think about the meaning and ways of calculating (proper fraction) ÷ (proper fraction). They will be able to carry out those calculations.	○ Think about the calculation to determine the area that can be painted with 1 dL of paint when ¾ dL of paint can paing ⅖ m^2. ○ Think about and explain the reason of the calculation.	[Interest] Students will become interested in the meaning and ways of calculating fraction ÷ fraction, and they try to make connections to their prior learning of calculations and properties of division. [Thinking] Students can think about ways of calculating fraction ÷ fraction based on the properties of division and proportional relationship, and they can represent those situations using double number lines and equations. [Abilities to compare, identify, and think]
L2		○ Think about ways of calculating ⅖ ÷ ¾. ○ Summarize the way to calculate (proper fraction) ÷ (proper fraction).	
L3	Students understand that simplifying fractions while calculating may calculation simpler. Students will understand how to calculate (whole number) ÷ (fraction) and division with mixed numbers.	○ Think about ways to calculate 9/14 ÷ 3/4. ○ Think about ways to calculate 4 ÷ 9/2. ○ Think about ways to calculate ⅔ ÷ 3⅕.	[Interest] Students recognize that simplifying fractions while calculating may make calculation simpler. [Skills] Students can calculate fraction ÷ fraction and fractions with mixed numbers.
L4	Students understand that division by a proper fraction will result in the quotient greater than the dividend. Students will understand ways to calculate with 3	○ Calculate 12 ÷ 1⅓ and 12 ÷ ⅔ and compare the quotients with the dividends. ○ Summarize the relationship that division by a proper fraction will result in the quotient greater than the dividend.	[Thinking] Using double number lines, students think about and explain the relationship between the size relationship between the dividend and the quotient depending on the size of the divisor. [Skills] Students can calculate

	fractions with both × and ÷.	○ Think about ways to calculate 3/4 ÷ 6/5 × 1/5.	with 3 fractions with both × and ÷. [Abilities to compare, identify, and think]
L5	Students understand that multiplication and division with mixture of fractions, decimal numbers and whole numbers may be easier to carry out when decimal numbers and whole numbers are expressed as fractions. Students can carry out those calculations.	○ Think about ways to calculate 0.3 ÷ ⅗. ○ Summarize ways to calculate × and ÷ with mixture of fractions, decimal numbers and whole numbers.	[Skills] Students can calculate × and ÷ with mixture of fractions, decimal numbers and whole numbers.
L6	Students will deepen their understanding of how to determine the necessary operation using double number line.	○ Given 4/7 m of hose that weighs 2/5 kg, think about the calculations needed to determine the weight of 1m of this hose and the length of 1kg of this hose using double number line.	[Thinking] Using double number line, students think about and explain which division is appropriate for the problem situation. [Abilities to compare, identify, and think]
② Times as much with fractions and multiplication·division (4 lessons)			
L7	Students understand that even when the relative quantity and the referent quantity are fractions, calculation to determine the ratio (how many times as much) is still division.	○ Think about ways to determine how many times as long as ½ m are 5/4 m and 3/8 m. ○ Summarize ways to determine how many times as much when the relative quantity and the referent quantity are fractions.	[Thinking] Think about and explain the equation, (referent quantity) × (ratio) = (relative quantity), based on the meaning of times as much and double number line representation. [Skills] Students can determine how many times as much using division even when the relative quantity and the referent quantity are fractions. [Abilities to compare, identify, and think]
L8	Students understand that even when the ratio (how many times as much) is a	○ Think about ways to determine 6/5 times as much and 3/5 times as much of 600 yen.	[Thinking] Thinking about and explain the equation, (referent quantity) × (ratio) = (relative quantity), based on the

	fraction they can determine the relative quantity using (referent quantity) × (ratio) = (relative quantity).	○ Summarize how to determine the relative quantity when it is fraction times as much as the referent quantity.	meaning of times as much and double number line representation. [Skills] Students can determine the relative quantity using the referent quantity and the ratio even when the ratio is a fraction. [Abilities to compare, identify, and think]
L9	Students understand that even when the ratio (how many times as much) is a fraction they can determine the referent quantity using (relative quantity) ÷ ratio.	○ Think about ways to find the original price when 900 yen is 5/3 of the original price. ○ Set up an equation using *x* and find the value for *x*.	[Thinking] Thinking about and explain the equation, (referent quantity) × (ratio) = (relative quantity), based on the meaning of times as much and double number line representation. [Skills] Even when the ratio is a fraction, students can represent the relationship of quantities using a multiplication equation with *x* and determine the relative quantity. [Abilities to compare, identify, and think]
L10	Students reason that the quotient represents the ratio when the divisor is considered as 1 using diagrams and equations. **Today's Lesson**	○ In a quotitive division situation with a remainder, think about the meaning of the fraction part of the quotient 17½.	[Thinking] Students think about and explain the meaning of the fraction part, ½, using double number lines. [Ability to reflect and apply]
③ Summary (1 lesson)			
L11	Ensure the learning of the content and consolidate students' understanding.	○ Work on practice problems.	[Skills] Students can solve problems applying what they have learned. [Knowledge] Students have mastered the content of the unit.

5 Today's Lesson

(1) Goals of the Lesson

Students reason that the quotient represents the ratio when the divisor is considered as 1 using diagrams and equations.

(2) Instructional Intentions

The aim of this lesson is for students to think about the meaning of fraction times as much by interpreting the meaning of the quotient and the remainder in a quotitive division problem with a fractional quotient.

The problem used to introduce division by fractions was, "We can pait ⅖ m^2 of wall with ¾ dL of pait. How many m^2 of wall can we pait with 1 dL of paint?" This is a probelm to determine the per-unit amount, and it is a partitive division situation. Because the answer can be shown as 2/5 (m^2) ÷ 3/4 (dL) = 8/15 (m^2), and it is straightforward to interpret the quotient.

In contrast, in today's lesson, we will discuss a quotitive division problem: "We have 1½ kg of ground meat. We are going to make ⅕-kg hamburger patties. How many hamburger patties can we make?" Since we are making ⅕-kg hamburger patties, there will be some ground meat left over. However, in the division calculation, there is no remainder. The quotient will be 7½, and interpreting the fractional part, ½, of the quotient is a challenge.

We decided to use the numbers so that the fractional part of the quotient to be ½. We felt that students may have easier time grasping the ratio of ½ as a "half" of a patty.

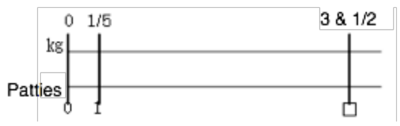
During the whole class discussion time, we will first discuss the meaning of the quotient. The anticipated responses are, "7 patties," "8 patties," "7 and ½ patties," and "7 patties and ½ kg will be left over." From this discussion, students will probably agree that "they can make 7 patties" and "there will be ground meet left over." However, they will also notice that both "patties" and "kg" are used in these responses. Therefore, we plan to ask, "Is it 7 patties and ½ kg left over or 7 patties and ½ of a patty?"

First, we will discuss what if "the remainder is ½ kg." Based on the relationship between the divisor and the remainder and calculation to check the result of division, we can verify that ½ does not mean the remainder of ½ kg. Next, by checking the results of division considering the quotient as 7½ patties, we get the correct dividend (the original amount of meat). However, because of the notation, "7½," some students may not understand that there will be meat left over or the amount of meat ½ represents." Therefore, we want to ask students what 7½ patties mean

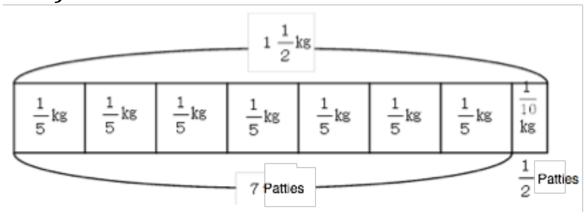
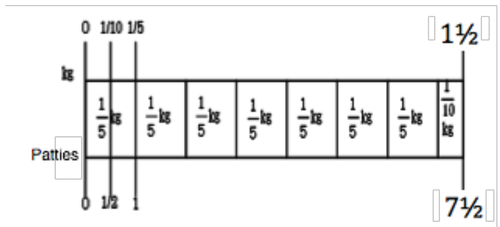
In order to help students grasp 7½ patties visually, we will use diagrams and double number line. From the diagram, if we keep making ⅕-patties, students can see that we can make 7patties, and the remaining part in the diagram represents ½ of a patty. Students then may say "half" or realize that it is "a half of a hamburger patty." We can summarize this as "½ of 1 hamburger patty (⅕ kg) is 1/10 kg." In other words, we want to emphasize that "½ is not ½ kg but ½ of ⅕ kg." Thus the answer to the problem for this lesson can be "7½ patties (7 ⅕-kg patties and ½ of ⅕-kg patty)" or "we can make 7 patties and 1/10 kg of meat ,which is ½ of ⅕-kg patty, will be left." Finally, from double number line, we want students to interpret the quotient as the ratio - "if we consider ⅕ kg as 1, 1½ kg will correspond to 7½" and "7½ times as much of ⅕ kg is a½ kg."

As a way to reflect on lessons, we have been asking students to write journals. At the end of the lesson, we would like to see entries that show the recognition of the importance of carefully interpreting the results of calculation and the way to see the quotient as the ratio such as following:
"I learned that ½ is not ½ kg left over. It means ½ (ratio) of 1 patty (⅕ kg)." "From double number line, I can see the relationship of the 2 quantities. If we consider ⅕ kg as 1, 1½ corresponds to 7½. If we consider ⅕ kg as 1, ½ corresponds to 1/10 kg." We would like students to realize the importance of examining the result of calculations like "If we make ⅕-kg hamburger patties from 1½ kg of ground meat, we can make 7½ patties" and consider quotients are ratios.

(3) Flow of the Lesson

min.	Main Learning Activity, Content Anticipated Responses	Points of Consideration Relationship to Research Theme
5 Grasp 5 Think	1. Understand the problem situation. We have 1½ kg of ground meat. We are going to make ½-kg hamburger patties. How many hamburger patties can we make? • Set up an equation from the problem situation and find the answer. • 1½ ÷ ½ = 3. We can make 3 patties. We have 1½ kg of ground meat. We are going to make ⅕-kg hamburger patties. How many hamburger patties can we make? 2. Independently solve the problem. • Set up an equation from the problem situation and find the answer. 1½ ÷ ⅕ = 15/2 = 7½ a) 7 patties b) 8 patties c) 7 and ½ patties d) 7 patties, ½ kg left over	• Display a quotitive division problem (whole number quotient). • Set up the equation using the double number line as necessary. • From the diagram, verify that we can make 3 patties. • Display a quotitive division problem (mixed number quotient). • Set up the equation using the double number line as necessary.  • Make sure that students will answer the number of hamburger patties we can make, not just doing the calculation.
30 Dig deeper	3. Compare and contrast the ideas. • Discuss the interpretation of answers (a) ~ (d). ○ (a) says 7 and a half, so we can make 7 patties. ○ (b) means we can make 7 patties that are ⅕ kg each and another small one with the leftover meat. So, we can make 8 patties. ○ (c) - Because the answer of calculation is 7½, 7 patties and ½ of a patty.	In the process of " lessons in which students feel the values of learning" ◎Disposition･Ability we want to nurture • (2) Ability to Compare, Identify, and Think ◎Students who are feeling the values learning • Tackle new problems using prior learning.

	○ The whole number part of (d) is the number of patties. The fraction part (½) is the remainder. So, we can make 7 patties and ½ kg of meat will be left. "What's the same? What's different?" ○ All of them says we can make 17 patties. ○ I think all of them are saying that there will be some ground meat left over. ○ There are different units. (d) sayd ½ kg but (c) says ½ patties. Not sure about (a) and (b).	• Explain the meaning of the quotient and the remainder in quotitive situations using diagrams. • Interpret each other's ideas. • Grasp that the quotient is the ratio when the divisor is considered as 1 usign diagrams (and equations). • Make sure that everyone understands that we can make 7 patties. ○ It is anticipated that the majority of students will support (d). Ask students to identify commonality and difference. Then ask, "Is it 7 patties and ½ kg of meant left over, or 7½ patties?"
	Let's think if it should be 7 patties and ½ kg left over, or 7 patties and ½ of a patty. • Verify that the remainder is not ½ kg. "Does ½ mean there will be ½ kg of meat left over?" "Let's think what if the quotient means 7 patties and ½kg left?" ○ ○ If ½ is the remainder, it will be greater than the divisor (⅕). We should be able to make more hamburger patties. ○ If we check the calculation with the quotient of 7 and the remainder of ½, it will be ⅕ × 7 + ½ = 19/10=1 9/10. So, we don't get the original amount. So, the remainder is not ½ kg. ○ If we check the calculation with 7½ as the quotient, it will be ⅕ × 7½ = 1½. It matches the dividend. ○ In the first question, the answer was a whole number, 3. We could make exactly 3 patties. This time, it is 7½, so we should have some ground meat left but... ○ 7 means 7 ⅕-kg patties, but what does ½ patties mean?	◎Strategies • Setting up the task • "Questions" "What's the same? What's different?" "What if the quotient means 7 patties and ½kg?" "What does 7½ patties mean?" ◎Support • Help students have ideas about possible solution approaches. • Write "questions" on the board. • Provide opportunities to interpret other students' ideas. ○ ○ Compare the size of the divisor and the remainder, or check the division calculation. Make sure students understand that the remainder does not mean ½ kg. ○ Based on the opening problem, students will think there will be left over meat . "Will there be leftover or not?" "What does 7½ patties mean?" Check using diagrams. ○ Allocate time for independent problem solving time.

5	• Think about the meaning of ½. "What does 7½ patties mean? Can you explain using a diagram (double number line)?"  ○ I drew a diagram. If I keep taking away ⅕ kg, I see that I can make 7 ⅕-kg patties. ○ I can see that there will be left over. I wonder if this part is ½ patties. ○ A half of a hamburger patty, so a half of ⅕ kg. ○ We can find the weight by ⅕ ÷ 2. It's 1/10 kg. ○ ½ is not ½ kg. It means ½ of ⅕ kg. If I calculate, ⅕ × ½ = 1/10. ○ 7½ patties mean we can make 7 ⅕-kg patties and a half of a ⅕-kg patty. ○ If we make ⅕ kg hamburger patties with 1½ kg ground meat, we can make 7½ patties. • Interpret the quantitative relationship from the double number line. ○ Each patty is ⅕ kg. 7½ patties will be 1½ kg. ○ In other words, if we consider ⅕ kg as 1, 1½ kg corresponds to 7½. ○ If we consider ⅕ kg as 1, what corresponds to 1/10 kg will be ½. ○ Instead of saying ½ kg left over, we should say we can make 7 patties and 1/10 kg, which is a half of a patty, left over.	○ By segmenting 1½ kg into ⅕ kg pieces, confirm that we can make 7 patties. ○ Ask students what kind of number is being represented by the ½ patties in the diagrm. ○ Based on "½ patties" organize the ideas "half," "a half of a patty," "a half of ⅕ kg," etc. ○ From calculations, determine the amount of leftover. ○ If necessary, display an enlarged diagram that shows the amount of 1 hamburger. ○ Remind students about the problem of determining "6/5 times as much of 600 yen." (⅕ in that context was ⅕ of 600 yen, not ⅕ yen.) ○ • Insert tick marks in the diagram so that it would look like double number lines. • When checking the answer compare with double number lines to review the ideas of fraction times as much and proportional relationships.  • Use students' journals to assess students' learning and the effectiveness of the lesson.

	4. Reflect on learning in this lesson. ○ ½ in 7½ did not mean there will be ½ kg leftover. It means ½ (ratio) of 1 hamburger (⅕ kg). ○ ○ I understood from the diagram and double number lines that if I consider ⅕ kg as 1, 1½ kg corresponds to 7½, and 1/10 kg corresponds to ½. ○ If we make ⅕-kg hamburger patties using 1½ kg of ground meat, we can make 7½ patties.	

(4) Observation Points

- Were the strategies to make the lesson a "lesson in which students feel the values of learning" effective?
  - ① Strategies to help students approach the lesson expectantly and with own questions
  - ② Strategies to encourage students to autonomously engage with the task, peers, and self
  - ③ Strategies to reflect on learning so that students can have the sense of accomplishment and satisfaction

(5) References --- omitted