June 29
Lesson 2R

University of Yamanashi Model Elementary School

Grade 3 Mathematics Lesson Plan

Division with Remainders: Utilizing Remainders

Teacher: Sayuri Kasai
Learners: 31 Students from Classroom 1
Lesson Location: Aogiri Hall
Discussion Location: Aogiri Hall

I. About the Unit

A 2012 report by the National Institute of Educational Policy Research examined the 4 year trends on students' performance on the National Assessment of Academic Ability[i] from 2007 to 2010. One of the areas where successful results have been achieved is the calculation of the four arithmetic operations with whole numbers, decimal numbers and fractions. The average success rate on questions on whole number multiplication and division was 89.3%. The report concludes that mastery of computational fluency is generally good. One of the challenges that still need to be addressed is "understanding the meaning of multiplication and division." This suggests that students can calculate fluently but unable to apply calculation appropriately because they do not understand the meaning of operations well enough. This unit is organized to address this challenge by examining the meaning of remainders and developing the disposition to utilize the idea of remainders in daily lives.

Division is introduced in Grade 3. Mathematically, division is the inverse operation of multiplication. Thus, we can define this way: if $b \times q = a$ (or $q \times b = a$), then $q = a \div b$. However, when division is introduced in Grade 3, we study division as a separate operation because we teach calculation through manipulation in concrete situations. For example, we learn that the following 2 situations can both be represented as "$12 \div 3 = 4$": if we give 3 items to each person, 12 items can be shared among 4 people (quotitive division), and if we share 12 items equally among 3 people, each person will receive 4 items (partitive division). The meaning of division studied in this unit remains unchanged. Whether we are calculating division without remainder like $12 \div 3$ or division with remainders like $14 \div 3$, the process of making groups of 3 or making 3 equal groups remain the same.

When calculating for quotients, we related division as the inverse operation of multiplication. By connecting the process of making equal groups to the basic multiplication facts, we found the quotient, q, by using the multiplication facts to find the number that satisfies $b \times q = a$ (or $q \times b = a$). For division examined in this unit, there is no natural number for q. Generally, division is dealt without considering remainders by using the relationship studied in Grade 5, $a \div b = a/b$ (a and b are integers and $b \neq 0$). However, the idea of division with remainders is supported by the following theorem: For any two integers a and b, there is a unique pair of integers q and r such that $a = b \times q + r$ and $0 \leq r < b$. Thus, by applying this theorem to $a = 14$ and $b = 3$, we know that there is a unique pair of integers, 4 and 2 so that $14 = 3 \times 4 + 2$. We represent this relationship by "$14 \div 3 = 4$ Rem. 2." Thus, the meaning of $14 \div 3$ is to find the greatest integer, $\square$, so that $3 \times \square$ is less than 14, and the remainder.

In teaching this unit, we want to help students concretely and visually grasp the numbers corresponding to "group size" and "number of groups," and distinguish them from "remainder" through activities of making equal groups. At the same time, we will incorporate activities in which students will reason and explain their ideas by connecting manipulation of concrete objects with mathematical expressions and equations.

Furthermore, division we have previously studied (without remainder) can be considered as the special case of the new division (with remainders) where remainder is 0. By integrating their previous knowledge into the new knowledge, it is hoped that students can solidify their understanding of division.

We will also incorporate problems that will require students to interpret the meaning of the remainder. In some problem situations, we cannot have a remainder while in other situations, we can simply ignore remainders. Students will be required to examine the results of the calculation in the context of the original problem situations and derive the appropriate answers to the problems.

II Goals of the Unit

Students will understand division with remainders. They will also deepen their understanding of division and be able to use division.

- Building on the prior knowledge of division without remainder, students will grasp the meaning of division with remainders and ways to calculate by making use of the relationship between multiplication and division operations and the processes of making equal groups. (Interest, Eagerness, and Attitude)
- Students can grasp division by integrating division without remainder and division with remainder. They can represent the meaning of division and the ways of calculation by using concrete objects, diagrams and/or mathematical expressions and equations. (Mathematical Way of Thinking)
- Students can calculate division with remainder – they can determine the quotients and the remainders. (Mathematical Skills)
- Students will deepen their understanding of division by knowing the meaning of the remainder and the relationship between the remainder and the divisor. (Knowledge and Understanding)

III Relationship between the research theme and this unit

1. About characters and abilities necessary for learning toward harmonious living

Mathematics is a discipline which is taught systematically. Therefore, it is important that each student develops his or her own questions. Students can then recognize each other's strength from "differences" in their ideas and experience "understanding." We consider this to be the learning toward harmonious living. Moreover, the engine of this process is "questions" students develop. Therefore, we want to help students develop the following creative reasoning ability:

- Ability to think independently and develop own questions in problem solving situations,
- Ability to compare and contrast own idea with those of others in problem solving situations.

In this unit, students should develop questions such as the following as we progress through the unit. The question at the beginning of the unit is, "Can we find the answer using division we have already learned?" The questions that should arise as students manipulate concrete objects and think about the meaning and methods of division calculation are, "Can we represent the process of manipulation using an equations?" and "Can we represent division with remainders using an equation?" As we explore ways to utilize division in our everyday situations, students should ask questions such as "Can we use division?" and "How can we make use of remainders?"

2. Dispositions for learning toward harmonious living

In teaching mathematics, I aim for a lesson in which students can develop a series of questions by carefully developing problems. Problems should arise from everyday situations, and by putting them on the mathematical stage, students can develop their own "questions." My goal is for students to experience "understanding" by thinking about their questions and comparing and contrasting their ideas with other students' ideas consciously and intentionally. For such lessons, it is essential that students have the disposition to generate and solve their own questions. For students to generate and solve their own questions, it is necessary that they can think logically. And, to think logically, students must be able to express their ideas clearly.

The emphases in this unit are understanding the meaning of division and appropriately using division to solve problems. Therefore, students will engage in activities of making equal groups and relating manipulation of concrete objects to diagrams and expressions/equations as they did with division without remainder. By engaging in these activities repeatedly, students can understand the meaning of "dividing." Moreover, instead of just calculating procedurally, we will incorporate problems in which students must interpret the results of calculations. Through those problems, students will develop the habit of examining the meaning of remainders.

As a way to assess students' creative reasoning ability, we will make use of students' notebooks. During this academic year, we have been exploring ways to create notebooks that will clearly reflect the format of teaching through problem solving, "individual problem solving → comparing and discussing solutions → reflection." Students have learned to not only recording their own ideas developed during the individual problem solving time but also recording their classmates' ideas shared during the comparing and discussing solutions stage of the lesson. Moreover, as students record their own and classmates' ideas, they try to record reasons and explanations for an idea using words, pictures, diagrams, and expressions/equations. Finally, by writing a journal entry, students can reflect and organize their reasoning so that they can make use of their own learning in future problem solving. By examining "individual problem solving," "comparing and discussing solutions" and "journal" components of students' notebooks, we can infer what students were thinking at each stage and how their thinking evolved during the lesson. By assessing how students thinking evolved, we want to examine how well students' creating reasoning ability has developed.

IV. Unit plan (10 lessons)

#	Goals	Learning Activities	Evaluation
(1)	Division with remainders (5 lessons)		
1	Students will understand how to calculate division with remainder in the case of single digit divisor and single digit quotient.	• Students will think about ways to calculate 14 ÷ 3.	• Building on the prior knowledge of division without remainder, students are trying to figure out ways to calculate division with remainders. (Interest, Eagerness, and Attitude) • Students can think about ways to calculate division with remainders based on their prior knowledge of division without remainder and explain using concrete objects, diagrams and/or equation. (Mathematical Way of Thinking) [Disposition to build on their prior learning.] [Disposition to use manipulation of concrete objects and diagrams to represent and communicate their ideas to others.]
2		• Students will learn that 14 ÷ 3 = 4 Rem. 2. • Students will understand the meaning of remainders.	
3	Students will understand the relationship between the divisor and the remainder.	• Students will investigate the relationship between the divisor and the remainder in the calculations of the form, □ ÷ 4.	• Students understand that the remainder will be less than the divisor. (Knowledge and understanding)
4	Students will understand division with remainder in the case of partitive division.	• After understanding the problem, students know that 16 ÷ 3 is the appropriate calculation and think about ways to calculate.	• Students can think about ways to calculate the quotient and the remainder of partitive division based on their prior knowledge of partitive division without remainder

			and explain using concrete objects, diagrams and/or equation. (Mathematical Way of Thinking) [Disposition to build on their prior learning.] [Disposition to use manipulation of concrete objects and diagrams to represent and communicate their ideas to others.]
5	Students will understand how to check their calculation in the case of division with remainders.	• Students will think about ways to check their calculation in the case of division with remainders.	• Students understand ways to check their calculation in the case of division with remainders.
(2) Problems that requires students to reason about remainders (3 lessons)			
1	Students will deepen their understanding of how to interpret remainders.	• After understanding the problem, students know that 23 ÷ 4 is appropriate and calculate. • Discuss whether or not the answer should be 5 because the results of the calculation is 5 Rem. 3. • Summarize that the answer will be Quotient + 1.	• Students understand how to process quotients and remainders appropriately based on problem contexts. (Knowledge and understanding)
2		• After understanding the problem, students know that 30 ÷ 4 is appropriate and calculate. • Although the remainder will be 2, discuss whether the answer should be the Quotient or Quotient + 1.	
3		• After understanding the problem, students know that 32 ÷ 5 is appropriate and calculate. • Although the results of the calculation is 6 Rem. 2, discuss what should be the answer based on the problem context of making groups.	

(3) Summary of the unit (2 lessons)			
1	Students solve problems by focusing on remainders. (Today's lesson)	• Determine the answer by listing all possibilities. • After understanding the problem, students will determine the appropriate calculation and calculate. • Determine the answer to the problem using the remainder.	• Students can identify patterns from problem contexts and explain their ideas using diagrams and expressions/ equations. [Disposition to build on their prior learning.] [Disposition to use manipulation of concrete objects, expressions/equations and diagrams to represent and to compare and contrast their own ideas with those of others.]
2	Students will create and solve problems using what they learned in this unit.	• Students will create their own problems. • Students will share and solve each other's problems.	Students can solve problems using what they learned in this unit. (Mathematical Skills)

V. Today's lesson

(1) Date Saturday, June 29, 2013 (10:00 – 10:45)

(2) Location University of Yamanashi Model Elementary School Aogiri Hall

(3) Goal of the lesson

- Students will recognize that problems may be solved by focusing on the remainders.

(4) Rationale of the lesson

The problem used in today's lesson does not involve key words/phrases that are often associated with division such as "share," "how many times," or "how many groups." Students must identify patterns from the problem situation and represent the identified patterns in a table and mathematical expressions with addition, multiplication and division in order to solve the problem.

In our class, every Wednesday is designated as the day we play together as a class. The games used on those days are chosen by the member of the Recreation Committee. The problem situation involves creating a circular chart for those games similar to the one that show various classroom duties that must be rotated. Suppose there are four different games, "dodge bee," "keidro," "dodge ball," and "kohri," and we play one game a week. We play "dodge bee" in week 1, "keidro" in week 2, "dodge ball" in week 3 and "kohri" in week 4. Then, in week 5, we go back to "dodge bee," in week 6, we play "keidro," in week 7, we play "dodge ball," and so on. The question will be to determine which game we will be playing in week 26. The goal is to represent the situation mathematically and find the answer.

The mathematical representation we expect in today's lesson is expressions/equations. One way to find the answer is to list the games we play through week 26 one by one. This method might be a comfortable one for some students because they know that the correct answer will be found for sure. However, as the number of weeks becomes large, it is a tedious method. At that point, they will be asked to think about "methods that can be accurate yet simple even when numbers become large." It is hoped that students will notice that the numbers (for a particular game) increases by 4, and some may try to use the multiplication table for 4's to find the answer.

If students make a table, it is likely that they will notice that for the fourth game, "kohri," the week numbers will be 4, 8, 12, …, that is, it matches the multiplication table for the 4's facts. Then, they will note that $4 \times 6 = 24$. Therefore, the game for week 26 should be the 2nd game from "kohri," or "keidro." When students make this observation, they will be encouraged to represent the idea "the 2nd from 24" using mathematical expressions/equations, in particular, $26 = 4 \times 6 + 2$. It is hoped that students will realize that this equation is the same form as the one we used to check the answers for division calculations. It is hoped that this observation will lead to another "question, " "Can we use division (or the remainder)?"

The week numbers for the 4th game, "kohri," will be those numbers that can be evenly divided by 4. Thus, the week numbers for the 1st game, "dodge bee," will have the remainder of 1 when divided by 4. Similarly, the week numbers for the 2nd game, "keidro," will have the remainder of 2 while the week numbers for the 3rd game, "dodge ball,"will have the remainder of 3. $26 \div 4 = 6$

Rem. 2. Therefore, the game with the week numbers with the remainder of 2 is “keidro.” Students should be able to experience the benefit that by focusing on the remainder, they can determine accurately which game will be played in which week no matter what the week number may be. This solution method leads to the idea of generalizability since it can be used “for any number.”

In order to generate the series of “questions,” students will be given time to listen to and make sense of their classmates’ ideas. It is anticipated that a variety of strategies will be presented during the comparing and discussing solutions stage of the lesson. By having students explain other students’ ideas or discussing unique features of their solution strategies, they might notice the differences between their own solution methods and other approaches and identify advantages of other strategies. Moreover, they may be able to incorporate those advantages to their own strategies in the future.

As a way to reflect on lessons, students have been keeping journals. Hopefully, there will be entries that reflect students’ recognition of the usefulness of mathematical representations such as “By listing the numbers (in a table), I was able to find the answer,” or “I learned that we can find the answer by calculation instead of listing all the numbers.” I also hope to see some entries that might suggest the disposition to use mathematics in dealing with everyday problems such as “I understood that the remainders can be useful to solve problems.”

(5) Flow of the lesson

Min	Content and Tasks	Instructional considerations/Relationship to the research theme
5	1 Understanding the problem ・ Notice the structure of the Recreation chart. What game will we be playing in week ☐? ○ We play kohri in weeks 8 and 12. ○ The week numbers for kohri are the multiplication table for 4's. Let's think about ways to find out which game we play in week 26.	・ Display the circular Recreation chart. ・ By displaying how 4 games are rotated from week to week, help students notice the pattern that a cycle of 4 games is completed in 4 weeks. ・ By discussing the week numbers for kohri (4, 8, 12, ...), help students realize that they might be able to use patterns to solve the problem.
10	2. Individual problem solving (a) List all week numbers up to 22. dodge bee 1 5 9 13 17 21 25 keidro 2 6 10 14 18 22 26 dodge ball 3 7 11 15 19 23 kohri 4 8 12 16 20 24 We will be playing keidro in week 26. (b) Use addition. From the table, I see that numbers for each game is increasing by 4. dodge bee 1 + 4 + 4 + 4 + 4 + 4 + 4 = 25 keidro 2 + 4 + 4 + 4 + 4 + 4 + 4 = 26 We get 22 with keidro. (c) Use multiplication. From the table, I see that the numbers for kohri are the same as the multiplication table for 4. The closest fact to 26 is 4 × 6 = 24. Since 26 is the 2nd from 24, the game must be "keidro." If we represent this in an equation, it will be: 4 × 6 + 2 = 26.	During the process of "learning toward harmonious living" ◎ Abilities to nurture ・ Ability to reason using what they have learned previously. ・ Ability to express their ideas by making connections among manipulation, expressions/ equations, and diagrams. ◎ Desired students responses ・ Students will list all numbers to find the answer. ・ Students will write a mathematical expression/ equation from the table and find the answer. ・ Students will interpret others' ideas from diagrams or expressions/equations. ・ Students will recognize a new solution strategy after listening to others' ideas. ◎ Strategy ・ Carefully design the learning tasks. ・ "Questions" "Can we find the answer more quickly and accurately?" "Can we use division?" "Do the numbers for keidro always have the remainder of 2?" ◎ Support ・ Help students think about ways to find the answer. ・ Organize the blackboard writing so that students can identify expressions/equations from a table. ・ Give students opportunities to interpret others' ideas.

25	(d) Use remainders. 26 ÷ 4 = 6 Rem. 2 dodge bee 5 ÷ 4 = 1 Rem. 1 keidro 6 ÷ 4 = 1 Rem. 2 dodge ball 7 ÷ 4 = 1 Rem. 3 kohri 8 ÷ 4 = 1 (Rem. 0) Of the four games, "remainder 2" will be keidro. 3. Comparing and discussing the solutions. ・Make a table showing all numbers up to 22 and find the answer. (a) ○ Week 26 will be keidro. ・ Discuss the solution that used multiplication? "Is there a quicker and accurate way to find the answer?" 4 × 6 + 2 = 26 (c) ○ + 2 means the 2nd from that game. So, it is keidro. ○ If we can use multiplication, we should be able to use division, too. ○ This equation looks like what we used to check the answers for division calculations. ・ Discuss the solution that used remainders. "Can we use division to find the answer?" 26 ÷ 4 = 6 Rem. 2 (d) ○ The numbers for kohri can be divided evenly. The remainder of 2 means it will be the 2nd from kohri. That will be keidro. ○ Do the numbers for keidro always have the remainder of 2? ・ Verify if the remainder method can be used with other week numbers. "Do the numbers for keidro always have the remainder of 2?" 22 ÷ 4 = 45 Rem. 2 18 ÷ 4 = 4 Rem. 2 14 ÷ 4 = 3 Rem. 2 10 ÷ 4 = 2 Rem. 2 6 ÷ 4 = 1 Rem. 2 2 ÷ 4 = 0 Rem. 2 ○ The numbers for keidro always have the remainder of 2. ○ Which game will have numbers with the	・ Start with idea (a). ・ Listen for the comment, "I think we can find the answer without listing all of the numbers." Lead to the "question," "Is there a quicker and accurate way to find the answer?" ・ Discuss the interpretations of the equation, 4 × 6 + 2 = 26, and help students realize that this equation is in the same form as the equations used to check the answers for division calculations. ・ By verifying the remainder method will work for keidro in weeks other than week 26, have students experience the relationship, "keidro = remainder 2.". ・ By focusing on the remainders for other games, help students appreciate the usefulness of using division.

	remainder of 1? ○ If we use remainders, we can find the answer quickly even if the number becomes large.	
5	4 Reflecting on the lesson ○ We can figure out which game we will be playing if we make a table. ○ We could use calculation to find the answer instead of writing all the numbers. ○ I learned for the first time that remainders can be useful to solve problems.	・ From journal entries assess students' learning and the effectiveness of the lesson.

(6) Assessment points

1. Was the "ability to nurture" today's lesson focused in alignment with the dispositions for learning toward harmonious living?
2. Was today's lesson (its organization, the choice of tasks, instructional approach, etc.) effective to nurture characters and abilities necessary for learning toward harmonious living?
3. Through today's lesson, did students "generate new ideas from comparing and contrasting own ideas with others'"? What were some of the new ideas?
4. Were students appropriately assessed and given appropriate support?

(7) References

（１） 国立教育政策研究所(2012)
「全国学力・学習状況調査の４年間の調査結果から今後の取り組みが期待される内容のまとめ ～児童生徒への学習指導の改善・充実に向けて～ 小学校編」

Translator's Note: This is the report written by the National Institute of Educational Policy Research. It is available (only in Japanese) from http://www.nier.go.jp/4nenmatome/.

（２） 文部科学省（平成 14 年８月）
「個に応じた指導に関する指導資料 ～発展的な学習や補充的な学習の推進～（小学校算数編）」
（３） 中村享史（1991）国土社
「算数 考える力をのばす教材」

[i] Since FY2007, the Ministry of Education, Culture, Sports, Science and Technology (MEXT) has carried out the National Assessment of Academic Ability in mathematics and Japanese for students in the sixth year of elementary school and the third year of lower secondary school. According to the National Institute of Educational Policy Research, "the assessment seeks to ascertain and analyze the academic abilities and learning patterns of schoolchildren throughout Japan and to investigate the outcomes of educational policies and programs, identify issues requiring attention, and achieve improvements therein."