Grade 12 Mathematics Lesson Plan

Lesson that will generate differential equations as mathematical models

Saturday, June 21, 2014 (11:10 – 12:00)
Mathematics 6 α R (7 male and 8 female students)
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attached to Tokyo Gakugei University
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1.   Research Question for the Lesson

        Today's lesson is a part of a series of lessons that try to address one of our goals for Grades 11/12 students, "In order to solver real-life problems, students can perform the processes such as formulating mathematical expressions, manipulating mathematical objects, interpreting the results of mathematical processes, and evaluating the outcomes." In particular, today's lesson will focus on the development of the following process: "In order to solve real-life problems, students can formulate differential equations as mathematical models, solve the equations, and interpret/evaluate the mathematical solutions." Since this lesson is the first time the students will be engaged in this process, goals of the lesson are for students to develop the idea of solving a real-life problem using a mathematical model of differential equations and to experience the way of thinking necessary to develop mathematical models. I believe the experience of using highly practical differential equations as mathematical models is an instance of a practical purpose of mathematics education.

        Today's lesson is not the lesson that "explains" differential equations and their applications. Rather, the goal is to make today's lesson as the lesson in which students will "generate" differential equations as mathematical models. That is because when students generate differential equations themselves as they examine a particular phenomenon, they can understand experientially under what circumstances differential equations are useful or the meaning of differential equations themselves.

        So, what should a lesson in which students generate differential equations as mathematical models look like? Differential equations are mathematical expressions that describe changes themselves. In such situations, it is necessary for a problem solver to hypothesize the status of change. How does one develop the idea of hypothesizing the status of change and describe the change mathematically? What strategies should a teacher employ so that students will develop such an idea? Those are the research questions addressed in today's lesson. I will describe specific strategies for today's lesson later, but I present three strategies as guidelines.

- Strategy 1: grounding questions in an investigation so that they will facilitate activities
- Strategy 2: reflecting on activities to extend the activities
- Strategy 3: using a mathematical content of "sequence" as a tool for students' investigation

Strategies 1 and 2 are something that must be considered in every lesson, while Strategy 3 is a particular strategy for a lesson whose aim is for students to generate differential equations. Previously, in the unit on sequences, students have learned to examine sequences by expressing a sequence in a recursive formula. Based on this knowledge, in today's lesson, I want to help students to develop the idea that, by considering a recursive formula as a difference equation, it describes the changes in adjacent terms of a sequence. Clearly, difference equations are useful themselves, but, in today's lesson, by shifting our attention from discrete changes to continuous change, it is hoped that students will generate differential equations.

Figure 1, the structure of a level-raising lesson, organizes the discussion above. Today's lesson aims to raise the level from the level of representing changes in a phenomenon using recursive formulas students have previously learned to the level of representing the changes using differential equations. To achieve this goal, a real-life phenomenon in which changes can be represented with recursive formulas will be prepared (Strategy 1). After students develop recursive formulas to represent the changes, we will re-interpret recursive formulas as difference equations (Strategies 3 and 2). In addition, by selecting the phenomena in which the changes need to be considered continuously, not discretely, (Strategy 1) we will discuss the need for shortening the unit time so that the level can be raised to consider differential equations.
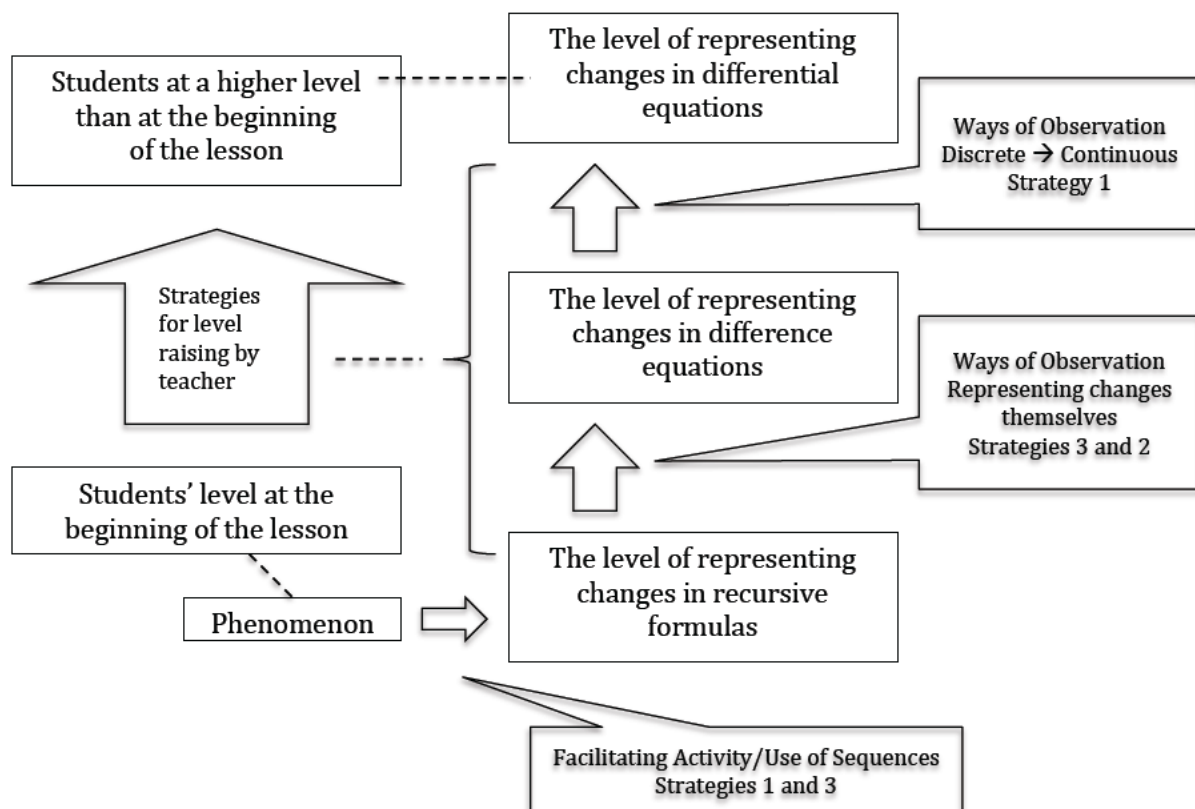

Figure 1 The structure of a level-raising lessons

In particular, today's lesson will make explicit the strategy to re-interpret recursive formulas as difference equations and the strategy to transform from difference equations to differential equations. We want to test whether or not these strategies will be effective.

2. About today's lesson
(1)   Research theme and difference/differential equations
       In today's lesson, I will use the following exploration task that utilizes the SIR model, the most traditional and foundational mathematical model for the spread of infectious diseases (for a more detailed discussion on the SIR model, see, for example, http://www.maa.org/publications/periodicals/loci/joma/the-sir-model-for-spread-of-disease-the-differential-equation-model).

There is an infectious disease with the probability of infection based on a single close contact at 1.8%. Suppose an infected person entered into a population of 100,000. In this population, an infected person makes, on average, 70 close contacts in a week. An infected person is diagnosed within a week of the contact at the probability of 99% and isolated from the population. Once a person recovers from the infection, he or she develops an immunity from the disease and will have no risk of additional infection.

You are the person in charge of public health in this population, and you want to encourage people to receive vaccination against this disease to avoid an outbreak.
(1) Make a simulation that shows the change in the number of infected people in this population if there was no vaccination.
(2) With the risk of potential side effects, it is not effective to mandate the vaccination for the entire population. Decide the minimum number of people who should receive the vaccination so that the risk of outbreak of this infection will be avoided.

Figure 2 Avoiding an Outbreak of Infection – today's task

This problem situation can be represented by the recursive formulas as follows. Let S_n be the number of people who have not been infected but susceptible to the infection (susceptible population), I_n be the number of infected people, and R_n be the total number of people immune to the infection (because they have recovered from the infection) and those who are isolated or died (removed population), each in week n. In addition, we assume that the total population is constant, $S_n + I_n + R_n = N$. If we assume that new infections only arise from the uninfected people in the population and the new removed population will come from the infected population, we can represent the situation in the following recursive formulas.

$$\begin{cases} S_{n+1} = S_n - S_n \times (I_n / N) \times 70 \times 0.018 \\ I_{n+1} = I_n + S_n \times (I_n / N) \times 70 \times 0.018 - I_n \quad \dots (2.1) \\ R_{n+1} = R_n + I_n \end{cases}$$

The graph below shows the results of the simulation based on these recursive formulas.
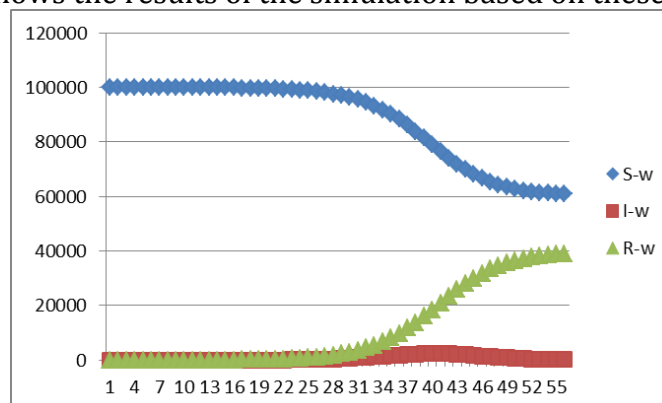

Figure 3 Results of the simulation

If we re-interpret the recursive formulas (2.1) as equations showing "weekly population change," we obtain the following set of difference equations.

$$\begin{cases} S_{n+1} - S_n = S_n \times (I_n / N) \times 70 \times 0.018 \\ I_{n+1} - I_n = S_n \times (I_n / N) \times 70 \times 0.018 - I_n \quad ...(2.2) \\ R_{n+1} - R_n = I_n \end{cases}$$

Then, we gradually reduce the size of unit time interval from "weekly" until unit time interval becomes "daily." Then, if we think about making the time interval approach 0, we obtain the following system of differential equations.

$$\begin{cases} dS(t)/dt = -S(t) \times (I(t)/N) \times 10 \times 0.018 \\ dI(t)/dt = S \times (I(t)/N) \times 10 \times 0.018 - (1/7)I(t) \\ dR(t)/dt = (1/7)I(t) \end{cases}$$

Moreover, if we consider $S \approx N$ at the beginning of an outbreak, the second equation above will become

$$dI(t)/dt = N \times (I(t)/N) \times 10 \times 0.018 - (1/7)I(t)$$
$$\approx 0.037 I(t).$$

Under the condition, $I(0) = 1$, this equation can be solved as $I(t) = e^{0.037t}$.

(2)   About mathematics

I believe that the use of the SIR infection model matches the intent of today's research lesson with the following three reasons.

First, to simulate the number of infected people, the need arises to consider not only the infected population (I) but also, at minimum, the number of the susceptible population (S). Therefore, it is necessary to consider the change of the infected population over a unit time to accommodate the movement of population from the susceptible population to the infected population. I expect students to represent these changes over time in some manners. Then, we will shift our focus on the changes themselves. We can anticipate that, as the representations of these changes are refined, difference/differential equations will be generated.

Next, by examining the initial stage of an outbreak, a first order linear differential equation, $dy/dx = ky$ will be generated. This form of differential equation is very concise and simple to solve, yet it is can be used to express a variety of phenomena. Thus, knowing this form of differential equation is useful in itself. Moreover, because this form of differential equation can be used to represent a variety of phenomena, we can use the assessment problem (see Appendix) to assess whether or not the goals of the research lesson are achieved. We will discuss the assessment problem later.

Finally, I believe that students can experience the practical use of differential equations through this exploration task. This situation is an actual example of situations in which differential equations are used. Students can experience how useful differential equations in real life can be as they consider mathematically ways to avoid or minimize the outbreak of an infection.

(3)   Unit Plan
        Differential equations are discussed toward the end of an approved textbook for Mathematics III as an extension topic. That is because the textbook discussion includes how to solve differential equations. However, the intent of today's lesson is not "solving differential equations" but "representing in differential equations" as a mathematical model. Thus, this unit is positioned as "an application of differentiation." Thus, it is assumed that students have already learned differentiation in Mathematics III.
        This unit will have the total of 4 lessons. Today's lesson will be the third lesson of the unit. It is anticipated that, by today's lesson, students have generated a system of recursive formulas (2.1). Moreover, by assuming $S_n \approx N$ at the beginning of an outbreak, the second equation in (2.1) can be transformed as $I_{n+1} = 1.26 I_n$. In other words, the number of people in the infected population, I_n, increases as a geometric sequence (discrete exponential function).

<div align="center">Table 1 Unit Plan</div>

Lesson	Activity	Goals
1	Independent problem solving (in groups) of the exploration task.	Students will be able to represent the phenomena in recursive formulas and run the simulation based on the formulas.
2	Sharing and discussion of the solution of the exploration task. Examination of the changes at the initial stage of an outbreak (discrete).	Students will be able to characterize the changes mathematically using recursive formulas.
3 (Today)	Examination of the changes at the beginning of an outbreak by re-interpreting the recursive formulas as difference equations (continuous).	Students will be able to describe the changes in the phenomena and generate differential equations.
4	Determine the vaccination rate based on the boundary conditions.	Students understand the process of mathematical modeling.

(4)   Specific strategies to help students generate differential equations as a mathematical model

        Of several instructional strategies developed for this unit, mainly the following two specific strategies will be employed in today's (lesson 3 of 4) lesson.

Strategy 3-(1): Ask students what they can observe about the weekly changes in the number of infected people. Purpose 3-(1): To help students re-interpret the recursive formulas to generate difference equations.

        As a part of the simulation to develop a preventative strategy, students will investigate the weekly increase in the number of infected people at the early stage of an outbreak. By asking about the increase instead of the number of infected people, weekly change will become students' focus. Although there are various ways to investigate the increase, in the end, I would like them to discover that "weekly increase

in the number of infected people is proportional to the number of infected people at the beginning of that week" based on the second recursive formula of (2.1) (with the assumption $S_n \approx N$).

Strategy 3-(2): Ask students if we can investigate changes in smaller time intervals than one week.
Purpose 3-(2): To generate differential equations from difference equations.

Based on the simulation using weekly changes in infected people, we can only discuss weekly preventative strategies. However, ideally, we would like to be able to investigate the changes in the number of infected people in much shorter time intervals, and ask students to think about ways to use shorter time units. In reality, we can probably make observations of the number of infected people every day, and we can use daily rates of change as approximations of instantaneous rates of change.

3. About students and their previous experiences

Students in Mathematics 6 α (Mathematics III) are students at an advanced standing, and they are very competent mathematically. They have many experiences of investigating phenomena in contexts mathematically, and they have the disposition to approach challenging problems independently. They are eager to tackle problems that pique their interests. Moreover, when the teacher asks a question to the whole class, several students are willing to share their ideas without being called upon. Moreover, they take other students' ideas and explanations seriously. One of the characteristics of the students in this grade is that they can write an in-depth reflection of their investigation.

In May of last year, 8 of the 15 students in the class experienced a 2-lesson Integrated Study unit in which they simulated the changes in population of living entities. In that particular unit, they developed the following ways of observing the population changes applying sequences. One way is to consider that the yearly increase in the population of sheep is proportional to the size of the population (discrete Malthusian model). The other is to consider that the yearly increase is proportional to the product of the total population and the number of survival population (discrete logistic model). However, we did not focus on the differences themselves since they have yet to learn about differentiation. Thus, what they considered was the yearly increase, not the rate of change in population over a unit time of one year. It is unknown how that experience will impact students' learning in the current unit. I plan to distribute those 8 students across the groups for this exploration.

4. Assessment Plan

After the completion of this unit based on the plan discussed above, the assessment task, Carbon-14 dating method, will be used to assess individual students' learning of the process, "In order to solve real-life problems, students can formulate differential equations as mathematical models, solve the equations, and interpret/evaluate the mathematical solutions." The evaluation rubrics are as follows.

Table 2 Evaluation Rubrics Perspective B. Processes and Reflection (B1)

0	Has not reached any of the levels below.
1-2	Students represented a concrete phenomenon using differential equations.
3-4	Students represented a concrete phenomenon using differential equations, and they can develop conclusions using mathematical manipulations.
5-6	Students represented a concrete phenomenon using differential equations, and they can develop conclusions using mathematical manipulations. Moreover, they can reflect on the process of manipulations and the conclusions and evaluate them.

As for the question of whether or not the level of activities students engaged as a class was raised will be assessed based on students' worksheets and also by keeping a record of students' remarks during the lesson.

5. Plan for Today's Lesson
(1) Goal of the lesson

Students will generate differential equations as a mathematical model and use them to represent changes in a phenomenon.

(2) Flow of the Lesson (condensed version)
T: teacher questions; S: Anticipated response by students; **O** teaching strategy

Time	Instructional content/main *hatsumon* and anticipated student responses	Instructional points of consideration
5	**1. Review of the previous lesson and presentation of a new task** T0: We learned that the change in the number of infected people at the early stage of an outbreak is a geometric sequence (discrete exponential function). **T1: In order to prevent an outbreak, we need to minimize the weekly increase of the number of infected people. Based on the simulation of an early stage of an outbreak, what can we say about the increase in the number of infected people?**	Ask the question so that the need for investigating the weekly increase in the number of patients.
10	**2. Independent problem solving in groups** S1-1: Analyze the data --- calculate the weekly increases and examine their differences and ratios.	

		S1-2: Analyze the graph --- calculate the weekly increases and make predictions based on the shapes of the graphs. S1-3: Analyze the general term --- using the equation for the general term, $I_n = 1.26^{n-1}$, generate the expression, $I_{n+1} - I_n$. S1-4: Analyze the recursive formula --- using the formula, $I_{n+1} = 1.26I_n \cdots ①$, generate the expression, $I_{n+1} - I_n$. ⭕ If S1-4's response does not come up, ask students if we can use the recursive formula. S1-5: It changes as a geometric sequence. / It is similar to the changes in the number of infected people. S1-6: It is 0.27 times as much as the number of infected people at the beginning of the week. S1-7: It is proportional to the number of infected people at the beginning of the week. $\left(I_{n+1} - I_n = 0.27I_n \cdots ② \right)$ ⭕ If S1-7's response does not come up, ask students what kind of a functional relationship might exist between the increase ($I_{n+1} - I_n$) and the number of infected people (I_n).	After some time is passed, call upon a group with S1-4 (if this idea has not come up, S1-3) which are analyzing equations and ask them what they can observe.
10	**3. *Neriage*** **T2: If the weekly increase in the number of infected people is proportional to the number of infected people at the beginning of the week, what happens to the increase when the number of infected people become 2, 3, … times as much?** S2: It also becomes 2, 3, … times as much. **T3: Conversely, let's verify that In will become a geometric sequence if we assume the weekly increase in the number of infected people is proportional to the number of infected people at the beginning of the week.** S3: If $I_{n+1} - I_n = aI_n$, then $I_{n+1} = (1 + a)I_n$. So, it is a geometric sequence. **T4: So, at this point, we can conclude that In we derived in the previous lesson becomes a geometric sequence, if we assume that "the weekly increase in the number of infected people is proportional to the number of infected people at the beginning of the week."**	Interpret what it means for the weekly increase is proportional to the number of infected people at the beginning of the week. Emphasize that the assumption is focused on the increase, $I_{n+1} - I_n$, not I_n, and by examining $I_{n+1} - I_n$ we can investigate the change.

20	**4. Further exploration**	Emphasize that the graphs
	T5: We have been examining the weekly change in the number of infected people. However, in order to make our preventative strategy as accurate as possible, we want to investigate how the number of infected people changes more closely. What can we do?	are dot plots and there are gaps in between points.
	S4: We should use a shorter time period as a unit than "weekly."	
	T6: What would be the ideal unit time interval?	
	S6-1: If we know the number of infected people at each instant, the graphs will become continuous.	
	S6-2: Is it necessary for us to know the number in each instant when we are working with the number of people?	
	S6-3: Realistically, daily numbers may be the best we can get.	
	S6-4: I think it is enough if we know daily numbers.	
	⭘ If the key words such as "continuous" or "instant" do not come up, remind students that I_n is a function of the number of weeks, n, thus discrete. Then ask students, ideally, if we can think of it as a function of time, t.	Make sure students understand that instantaneous changes are idealized notion.
	T7: Let's first explore the changes in the number of infected people at the beginning of the outbreak. How will the recursive formulas change?	We can anticipate that it will be an exponential function.
	S7-1: I think we can use 10 for the daily number of close contacts.	
	S7-2: I don't think the proportion of susceptible people who will be infected by a contact will remain the same.	Re-confirm the conditions. The rate of removal may be difficult to figure out. If the whole class discussion does not develop, give students to discuss it in groups.
	S7-3: If we consider I_n represents the number of infected people on the nth day, $$I_{n+1} = I_n + N \times \frac{I_n}{N} \times 10 \times 0.018 - (1/7)I_n \fallingdotseq 1.037 I_n \cdots ③$$	
	T8: What do we need to do if we consider $I(t)$ as a function of t and examine the change continuously?	
	S8-1: We need to shorten the unit time interval even further from "daily."	
	S8-2: I think we can even think in terms of "hourly."	
	S8-3: Maybe we can express the intervals as h or Δt, we can take the limit of the function.	

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S8-4: Maybe we can consider that the instantaneous rate of increase in the number of infected people is proportional to the number of infected people at that moment.

⭕ If S8-4's idea does not come up, remind the students that we have already seen the situation of making the intervals (the amount of increase) approach 0, and ask them how we represented it.

T9: Based on the recursive formula ③ and if Δt << 1, then, lets' represent the relationship between $I(t + \Delta t)$ and $I(t)$ and consider what happens $\Delta t \to 0$.

S9-1: $I(t + \Delta t) = I(t) + N \times \frac{I(t)}{N} \times 10\Delta t \times 0.018 - \left(\frac{1}{7}\right)\Delta t I(t) \cdots④$

If we transfer $I(t)$ to the left hand side and letting $\Delta t \to 0$,

$dI(t)/dt = 0.037I(t)$.

S9-2: If we assume that the instantaneous rate of increase of the number of infected people is proportional to the number of infected people at that moment, we can say $dI(t)/dt = 0.037I(t)$.

⭕ If S9-1's idea does not come up, re-interpret the left hand side of equation ③, $I_{n+1} - I_n$ as the average daily rate of change, and ask students how to express the instantaneous increase in the number of infected people.

⭕ Even if S8-4's response does not come up in response to T-8 above, make sure we interpret equation ④ as done by S9-2.

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Give students time to discuss this in groups if necessary.

Ideally, S8-4's idea would come up, but if it does not, make use of the idea like S8-3's and move the discussion forward.
If any group uses an time interval unit that is less than 1 hour and write a recursive formula, we can make use of it to think about Δt.

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5. Summary of the lesson and the task for the next lesson
T9: If we mathematically represent the assumption, "the rate of change in the number of infected people is proportional to the number of infected people," it will be $dI(t)/dt = aI(t)$. In this particular problem situation, we could express it as $dI(t)/dt = 0.037I(t)$. Therefore, $I(t)$ is a function whose derivative will be (approximately) 0.037 times as much as the function itself. In the next lesson, let's try to find the function that satisfies this condition.

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The label, "differential equation," will be given after the conclusion of the exploration in the next lesson.

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Appendix 1: Assessment Problem

In archeology, Carbon-14 dating method is used to determine the age of ancient clay pots or ancient remains. The nuclei of radioactive Carbon-14 are naturally unstable, and, without any external influence, they emit radioactive rays and disintegrate into different nuclei at a constant rate over a fixed time. Carbon-14 in the atmosphere is created by the cosmic rays, and its concentration is virtually constant. The concentration of Carbon-14 in living organisms also remains constant, because of photosynthesis in plants and food chains in animals, while they are still alive. However, once the living organism dies, Carbon-14 in the body will continuously disintegrate without absorption of any additional Carbon-14 from outside.

A seed was discovered in a clay pot. It was determined that the number of Carbon-14 nuclei is 4.2×10^{10}. It is known that a seed of the same plant today contains 6.0×10^{10} Carbon-14 nuclei. We want to estimate about how many years ago the clay pot was being used.

(1) The time it takes for the number of nuclei to become ½ of the original number is called "falf-life." Show that the half-life of Carbon-14 is constant.

(2) It has been measured that the half-life of Carbon-14 is approximately 5730 years. About how many years ago can we estimate that this clay pot was being used?

Graph 2-1 Assessment problem, "Carbon-14 dating method"

Appendix 2: Worksheet

The following worksheet was handed out to the students when the task was initially posed.

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TGUISS6 α Date   Class   Name

Let's prevent an outbreak!

       Suppose a person who is infected by a particular infectious disease entered into a population of 100,000. When an uninfected person comes into a close contact with an infected person, there is a probability of 1.8 % that the uninfected person will be infected. A person who was infected by this disease can infect other people for 7 days after he or she is infected. When a person becomes no longer infectious, i.e., when he or she is recovered, the person develops immunity from this disease. Finally, in this population, an infected person makes, on average, 70 close contacts in a week.

       You are the person in charge of public health in this population, and you want to encourage people to receive vaccination against this disease to avoid an outbreak. When you conducted a survey, it was found that, in this population, an infected person makes, on the average, 70 close contacts with others.

(1) Make a simulation that shows the change in the number of infected people in this population if there was no vaccination.

(2) With the risk of potential side effects, it is not effective to mandate the vaccination for the entire population. Decide the minimum number of people who should receive the vaccination so that the risk of outbreak of this infection will be avoided.